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**ADDENDUM**

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**Abstract**
The stress contribution to the inertia of a spherically symmetric charged particle is calculated for a generic stress tensor of spherical symmetry. It is found that it is equal to the result for the isotropic pressure case, which has been previously calculated (Medina R 2006 *J. Phys. A: Math. Gen.* **39** 3801–16).

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In a previous paper [1] the essential role played by the inertia of stress in the self-interaction of a charged particle was shown. In the rest frame the stress tensor of a body with spherical symmetry has the following form,

$$\mathbf{P} = P_r \hat{\boldsymbol{r}}\hat{\boldsymbol{r}} + P_t(\mathbf{I} - \hat{\boldsymbol{r}}\hat{\boldsymbol{r}}), \tag{1}$$

where the radial stress $P_r$ and the transverse stress $P_t$ are functions of the radius $r$, $\mathbf{I}$ is the identity tensor and $\hat{\boldsymbol{r}}$ is the radial unit vector.

In the paper cited above the stress contribution to the inertia was determined in the special case of an isotropic pressure $P = P_t = P_r$, and it was stated without proof that the same result is obtained for any stress tensor given by (1). Here we show that it is indeed so.

It was shown that the stress contributions to the energy and momentum of a particle moving with velocity $\boldsymbol{v}$ are

$$U_S = m_P \gamma v^2 \tag{2}$$

and

$$\boldsymbol{P}_S = m_P \gamma \boldsymbol{v}. \tag{3}$$

The mass of stress $m_P$ is given by the following integral that must be calculated in the rest frame:

$$m_P = \frac{1}{c^2}\int \mathrm{d}^3x\, P. \tag{4}$$

The pressure equilibrates the electrostatic force density $\boldsymbol{f} = \rho \boldsymbol{E}$. The pressure is obtained from the equilibrium equation $\nabla P = \boldsymbol{f}$, and it is found that the effective mass of pressure is proportional to the electrostatic energy $U_e$,

$$m_P = -\frac{1}{3c^2} U_e. \tag{5}$$

In the general case the energy density of stress is $P^{00}$ and the momentum density is $c^{-1} P^{0i}$, where $P^{\mu\nu}$ is the stress 4-tensor. As the stress 4-tensor is orthogonal to the 4-velocity $u^\mu$, i.e. $P^{\mu\nu} u_\nu = 0$, the energy and momentum contributions are

$$U_S = \frac{1}{c^2} \int \mathrm{d}^3 x\, \boldsymbol{v} \cdot \mathbf{P} \cdot \boldsymbol{v} \tag{6}$$

and

$$\boldsymbol{P}_S = \frac{1}{c^2} \int \mathrm{d}^3 x\, \mathbf{P} \cdot \boldsymbol{v}. \tag{7}$$

Here $\mathbf{P}$ is the stress tensor in the laboratory frame.

In the frame in which the particle moves with velocity $\boldsymbol{v}$, the two tensors that appear on the right-hand side of (1) are

$$\mathbf{I}_{\text{lab}} = \mathbf{I} + (\gamma^2 - 1)\hat{\boldsymbol{v}}\hat{\boldsymbol{v}} \tag{8}$$

and

$$(\hat{\boldsymbol{r}}\hat{\boldsymbol{r}})_{\text{lab}} = [\hat{\boldsymbol{r}} + (\gamma - 1)(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{v}})\hat{\boldsymbol{v}}][\hat{\boldsymbol{r}} + (\gamma - 1)(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{v}})\hat{\boldsymbol{v}}], \tag{9}$$

where $\hat{\mathbf{r}}$ is the radial unit vector in the rest frame. The temporal components of $P^{\mu\nu}$ are then

$$P^{00} = c^{-2} \boldsymbol{v} \cdot \mathbf{P} \cdot \boldsymbol{v} = (v/c)^2 \gamma^2 [P_t + (P_r - P_t)(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{v}})^2] \tag{10}$$

and

$$\mathbf{P} \cdot \boldsymbol{v} = P_t \gamma^2 \boldsymbol{v} + (P_r - P_t)(\hat{\boldsymbol{r}} \cdot \boldsymbol{v})\gamma[\hat{\boldsymbol{r}} + (\gamma - 1)(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{v}})\hat{\boldsymbol{v}}]. \tag{11}$$

Using (10) and (11) in (6) and (7), changing the variables of integration to the rest frame coordinates, which gives a factor $\gamma^{-1}$, and taking into account the fact that

$$\frac{1}{4\pi} \int \mathrm{d}\Omega\, \hat{\boldsymbol{r}}\hat{\boldsymbol{r}} = \frac{1}{3}\mathbf{I} \tag{12}$$

one obtains that the energy and momentum contributions of stress are given by (2) and (3), but with a stress mass given by

$$m_p = \frac{1}{3c^2} \int \mathrm{d}^3 x\, (2P_t + P_r) \tag{13}$$

instead of (4).

In the general case the equilibrium equation for the stress tensor is

$$\boldsymbol{f} = \nabla \cdot \mathbf{P}, \tag{14}$$

which for a stress given by (1) and for a particle with charge density $\rho(r)$ can be written as

$$\rho E = \frac{\mathrm{d}P_r}{\mathrm{d}r} + 2\frac{P_r - P_t}{r}. \tag{15}$$

Multiplying (15) by $4\pi r^3$ and integrating one obtains

$$\begin{aligned} \int_0^\infty \mathrm{d}r\, 4\pi r^3 \rho E &= \int_0^\infty \mathrm{d}r\, 4\pi r^2 \left[ r\frac{\mathrm{d}P_r}{\mathrm{d}r} + 2P_r - 2P_t \right] \\ &= 4\pi r^3 P_r|_0^\infty - \int \mathrm{d}^3 x\, (2P_t + P_r) \\ &= -3m_P c^2. \end{aligned} \tag{16}$$

On the other hand, $E(r) = Q(r)/r^2$, where $Q(r)$ is the charge enclosed in a sphere of radius $r$,

$$Q(r) = 4\pi \int_0^r \mathrm{d}y\, y^2 \rho(y). \tag{17}$$

Thus, the left-hand side of (16) becomes

$$\begin{aligned}\int_0^\infty \mathrm{d}r 4\pi r^3 \rho E &= \int_0^\infty \mathrm{d}r\, 4\pi r^2 \rho \frac{Q}{r} \\ &= \int \mathrm{d}Q \frac{Q}{r} \\ &= \frac{1}{2} \int \mathrm{d}Q^2 \frac{1}{r} \\ &= \frac{1}{2} \frac{Q^2}{r}\bigg|_0^\infty + \frac{1}{2} \int_0^\infty \mathrm{d}r \frac{Q^2}{r^2} \\ &= \frac{1}{8\pi} \int \mathrm{d}^3 x \left[\frac{Q}{r^2}\right]^2 \\ &= U_e. \end{aligned} \tag{18}$$

Therefore, also in the general case the effective mass of stress is given by (5).

## References

[1] Medina R 2006 *J. Phys. A: Math. Gen.* **39** 3801–16